Boundary Condition for Rocket Motor Stability

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Recent advances in the understanding of the behavior of acoustic admittance surfaces are applied to the description of the oscillating mass flux from a burning solid propellant. Whereas current mathematical formulations describing the response of a rocket motor cavity to periodic disturbances require a boundary condition describing the mass flux response of the combustion zone at a time-independent boundary, the true interface between the motor cavity and the combustion zone is an oscillating surface where acoustic energy is transmitted as *p-v* work. Using a surface-oriented coordinate system, the boundary condition at the unsteady interface is transformed to the steady mathematical boundary for application to the motor stability analysis. The derived boundary condition includes two additional terms involving steady cross-flow coupling and steady flux acceleration that are not present in the usual admittance boundary condition. The growth coefficient for tubular grain motors predicted using this boundary condition differs from standard predictions and is dependent on the steady flow description.

Introduction

COMBUSTION instabilities have hindered the development of rocket motors since their inception. Characterized by the presence of periodic pressure oscillations in the rocket motor and the nozzle, combustion instability may result in excessive mechanical stresses and undesirable vibrations in various system components and abnormally high heating rates at the burning surface. Since these consequences can, in turn, lead to system and/or mission failure, much research has been devoted to developing analytical procedures that can predict the stability characteristics of a rocket motor.

The frequency of oscillation in unstable operation is generally (at least initially) close to a natural acoustic mode of the rocket motor cavity (assuming rigid boundaries) and a linear or acoustic analysis is often sufficient to identify the stability limits of the motor.[1] Such acoustic or small-disturbance analyses have been successfully applied to a variety of aeroacoustic problems, including reduction of turbofan engine noise, jet exhaust noise, and general studies of sound attenuation in ducts.

The purpose of this paper is to draw attention to some recent advances in aeroacoustics leading to a better understanding of the behavior of an admittance surface in the presence of a steady flow and to show how this formulation applies to the admittance boundary condition of a burning propellant. A brief history of the development of the admittance boundary condition for an impenetrable surface will be presented and the analysis will be extended to consider an admittance surface with normal mass flux and cross flow. The results will be applied to the standard equations for predicting the rate of growth (or decay) of disturbances in a rocket motor and it will be shown that the final result is dependent on the steady flow model.

Linear Analyses

Linear analyses consider the properties in the volume of interest (i.e., rocket motor cavity, turbofan inlet duct, etc.) to be composed of a steady or mean part and a small harmonically fluctuating part. For example, the fluid velocity and pressure are

$$U^* = \bar{U}(x) + \epsilon u(x) e^{i\omega t} \quad (1)$$

$$p^* = \bar{p}(x) + \epsilon p(x) e^{i\omega t} \quad (2)$$

where x is the position vector.

Equations (1) and (2) are substituted into the appropriate governing equations (i.e., conservation of mass, momentum, etc.) and the terms grouped as terms of orders 1 and ϵ. Terms of order 1 are identifically satisfied by the steady flow; thus, the acoustician is concerned with the solution of the partial differential equations consisting of terms of order ϵ. These acoustic equations are to be solved over a finite volume V, consisting in this case of the motor cavity, and enclosed by a boundary surface S with a unit normal n pointing into the motor cavity. The effects of the external environment (i.e., rigid walls, combustion zone, and nozzle) on the motor cavity are introduced through appropriate boundary conditions. For rigid walls that represent a gas/solid interface, the physical location of the boundary surface is unambiguous. However, the boundary surface adjacent to the propellant (and the nozzle entrance) is a gas/gas interface and the proper physical location of this boundary is not obvious. Introduction of a boundary at this location is appropriate, since the gas in the motor cavity is usually assumed to be nonreacting with a multidimensional, nonuniform steady velocity, while the gas in the combustion zone is assumed to have a uniform steady velocity. Thus, it is appropriate to consider these two regions (motor cavity and combustion zone) as consisting of acoustically different media and the location of the boundary defines the physical separation of the different acoustic media. In practice, the combustion zone is assumed to be vanishingly thin and the boundary is placed at the propellant surface with no loss in generality. It is noted that the external environment (combustion zone and nozzle) are assumed to be acoustically "stiffer" than the motor cavity so that the influence of the environment can be described by an acoustic admittance.

Mathematically, the problem is formulated so that the motor cavity volume V_0, boundary surface S_0, and unit normal n_0 are independent of time.[1] For reasons to be discussed later, it is convenient to define the "physical interface" S as the surface that physically separates the regions of different acoustic media and the "mathematical boundary" S_0 as the time-independent surface surrounding the motor cavity. Since boundary conditions can be derived at the physical interfaces,

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it is clearly appropriate to select the various sections of the mathematical boundary to coincide with the physical interfaces.

The natural acoustic modes of the motor cavity are determined by solution of the acoustic equations subject to the rigid wall boundary conditions

$$\bar{U}\cdot n = 0 \qquad \text{on } S = S_0 \tag{3}$$

and

$$u\cdot n = 0 \qquad \text{on } S = S_0 \tag{4}$$

The stability limits are predicted by replacing the rigid wall conditions with a steady normal flux at the propellant surface

$$\bar{U}\cdot n = \bar{c}\bar{M}_b \qquad \text{on } S = S_0 \tag{5}$$

and an admittance or soft wall condition

$$u\cdot n = A_b p \qquad \text{on } S = S_0 \tag{6}$$

at the propellant surface, and determining whether the amplitude of small disturbances near the natural acoustic modes increase (unstable) or decrease (stable) with time.[1] Note that other influences such as the nozzle and particulate damping may be involved, but will not be referred to in this paper.

In practice, the stationary physical interface of the rigid walls used in determining the natural modes [Eq. (4)] is also used as the mathematical boundary for determining the stability using Eq. (6). However, Morse and Ingard[2] point out that the acoustic pressure acting on an admittance surface (physical interface) tends to make it move. Since the physical interface has a displacement, its location is not independent of time. (Note that this displacement is in response to the pressure fluctuations and is not related to the recession of the propellant surface.) Thus, while the mathematical boundary must be a stationary surface, the actual physical interface of the acoustic region (where it would be appropriate to specify a boundary condition based on physical compatibility) is not stationary. Although the difference between the location of the physical interface S and the mathematical boundary S_0 is small, the difference is of the order ϵ and must not be ignored.

The combustion zone and the three representations used in various analyses of the motor cavity are illustrated in Fig. 1. Figure 1a illustrates the stationary surface S_0 located near the burning surface of the propellant. In Fig. 1b, the combustion zone is replaced by a rigid media for determination of the natural modes of the motor cavity. In the absence of disturbances (Fig. 1c), the combustion zone and motor cavity are separated by the identical surfaces S and S_0. In the presence of disturbances (Fig. 1d), the physical interface S oscillates, while the mathematical boundary surface S_0 remains fixed.

Rather than revise the mathematical approach, it is simpler to formulate the boundary condition to include the effects of the interface displacement. For the harmonic time dependence assumed in Eqs. (1), the rigid wall interface, the undisturbed interface ($\epsilon = 0$), and the time-averaged location of the physical interface are equivalent and represent the appropriate stationary mathematical boundary S_0 for the stability analysis. (See Fig. 1.) However, the boundary condition at the physical interface cannot simply be applied at the stationary boundary, but must be transformed to S_0 in a manner consistent in retention of terms of order ϵ. It will be shown that the use of this revised boundary condition alters the predicted stability limits of the motor cavity.

Acoustic Boundary Conditions

Acoustic or small-disturbance analyses have been successfully applied to a variety of aeroacoustic problems, including reduction of turbofan inlet noise, jet exhaust noise, and machinery noise. A major consideration in many of these analyses has been the formulation of the proper boundary conditions.

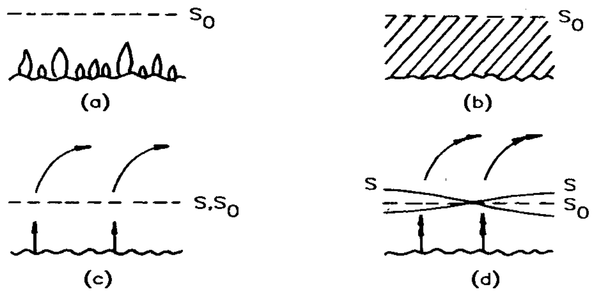

Fig. 1 Boundary interface models for the combustion zone: a) combustion zone, b) rigid wall interface, c) undisturbed physical interface, d) disturbed —— and time-averaged interface ——.

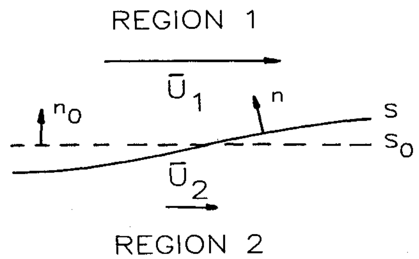

Fig. 2 General acoustic boundary separating two acoustic regions.

The interface between two acoustic regions shown in Fig. 2 represents a typical acoustic boundary. The solid line represents the physical interface or deformed surface S with unit normal n, while the dashed line represents the time-averaged or mathematical boundary S_0 with unit normal n_0. Region 1 contains a fluid with a steady velocity $\bar{U}_1$ and properties with subscript *1*, while properties in region 2 are similarly denoted by the subscript *2*. As before, regions 1 and 2 are assumed to have different steady velocities; thus, it is appropriate to consider the fluids as acoustically different media.

Early analyses[3] of the reflection and refraction of sound waves impinging obliquely upon the boundary of a high-speed gas jet ($\bar{U}_1 \gg \bar{U}_2$) ignored the differences between boundary locations and used the boundary conditions

$$\bar{U}_1\cdot n_0 = \bar{U}_2\cdot n_0 \qquad \text{on } S_0 \tag{7}$$

$$u_1\cdot n_0 = u_2\cdot n_0 \qquad \text{on } S_0 \tag{8}$$

This boundary condition is referred to as "continuity of normal velocity"[4] and is based on the kinematic boundary condition[5] of continuum fluid mechanics. If region 2 represents a passive, sound-absorbing material such as an acoustic duct liner, $\bar{U}_2 = 0$ and the mathematical boundary S_0 is taken to be an admittance surface. If the admittance A of the surface is defined by

$$u_2\cdot n = Ap_1 \qquad \text{on } S_0 \tag{9}$$

then substitution of Eq. (9) into Eq. (8) gives Eq. (6).

Concentrating on the formulation of a boundary condition at an interface of relative motion, Miles[6] and concurrently Ribner[7] postulated a principle referred to as "continuity of particle displacement." This principle requires that the displacement of a fluid "particle" at the physical interface S in region 1 must be equal to the displacement of a particle at the interface in region 2. These works represent the first recognition that the deformation of the interface is of order ϵ

and thus is an important factor in formulating the boundary condition.

If region 2 represents an acoustic liner, the interface S is an admittance surface and Eq. (6) is the proper boundary condition if there is no steady flow ($\bar{U}_1 = 0$) in region 1. However, when a steady flow exists in region 1, Ingard[8] applied the principle of particle displacement and obtained the boundary condition

$$u_1 \cdot n_0 = Ap_1 - (iA/\omega)\bar{U}_1 \cdot \nabla p_1 \qquad \text{on } S_0 \tag{10}$$

For some time there was considerable debate in the literature concerning Eqs. (6) and (10). Reference 4 presents a history of various attempts (comparison with experiment, asymptotic expansions, etc.) to determine the correct principle. Various expansions and computational procedures[9] convinced most duct acousticians that Eq. (10) was the correct boundary condition, but the principle of "continuity of particle displacement" was (and still is) considered a rather unusual foundation for a boundary condition in continuum fluid mechanics.

More recently, Taylor[10] introduced a derivation for a boundary condition at the surface of an oscillating body immersed in a steady flow. This condition was derived in a frame of reference in which the body generates a sound field by oscillations about a mean position. Myers[11] extended this approach to the more generalized case of the boundary condition on the acoustic perturbation velocity normal to a surface in a steady flow. The body surface is impermeable, but is acoustically deformed by an incident sound field. Myers showed that the principle of continuity of normal velocity was indeed the proper foundation for the boundary condition, but it must be applied at the physical interface S. The required boundary condition at the mathematical boundary S_0 is obtained by a transformation between surface-oriented coordinate systems using a series expansion that is consistent in the retention of terms of order ϵ. This procedure produces the boundary condition

$$u_1 \cdot n_0 = Ap_1 - \frac{i\bar{U}_1}{\omega} \cdot \nabla (Ap_1) + \frac{ip_1 A}{\omega} n_0 \cdot (n_0 \cdot \nabla \bar{U}_1) \qquad \text{on } S_0 \tag{11}$$

Myers also proves that the principle of continuity of particle displacement is not correct in general. In cases where the undeformed boundary surface S_0 is planar and the steady flow is everywhere parallel to the boundary plane, the last term in Eq. (11) vanishes and Eqs. (10) and (11) are equivalent. Thus, while the physical arguments behind Eq. (10) are in error, the results of Refs. 6-8 and the proofs of Ref. 9 are still valid.

Formulation of the Boundary Condition

The acoustic boundary condition for an impermeable admittance surface was derived by Myers[11] in terms of a curvilinear coordinate system. The analysis presented here follows the analysis of Myers, with two exceptions. The behavior of an impermeable admittance surface can be described entirely by the motion of physical interface without consideration of the properties within the acoustic liner (region 2). Since the physical interface or admittance surface at the edge of the combustion zone must pass a steady flow, it is appropriate to pose the boundary condition as a mass flux condition at the physical interface and consider the motion of the interface to be the motion of "acoustic particles" at the interface. These acoustic particles are not disturbed by the steady flow, but are displaced by fluctuating forces and thus define the physical interface. Second, for simplicity, the boundary condition will be derived in terms of a simple two-dimensional coordinate system appropriate to the axisymmetric tubular grain rocket motor. It will be shown that the kinematic portion of the mass flux term is equivalent to the impermeable boundary condition, so that the more general expression for a curvilinear coordinate system [Eq. (11)] can be applied.

The mathematical boundary surface and the physical interface are shown in Fig. 3. The dashed line represents the stationary mathematical boundary surface S_0. A coordinate system is fixed to the stationary surface so that n_0 is the unit vector in the normal or y direction and a_0 is the unit vector in the axial or x direction. The equation for the mathematical boundary is $y = 0$.

The solid line in Fig. 3 represents the physical interface that separates the two acoustic regions at any instant. This surface is described by the equation

$$y = \epsilon g(x) e^{i\omega t} \tag{12}$$

where terms of order ϵ^2 and higher have been neglected. The function g represents the displacement of the surface caused by disturbances (thus the possibility of higher-order terms) which must be determined in the course of the analysis. It is clear from Eq. (12) that the mathematical boundary ($y = 0$) is also the time-averaged location of the physical interface.

Combustion Zone Analysis

Region 2 represents the environment external to the motor cavity at the propellant surface, including the combustion zone, the propellant, the motor casing, etc. However, the mass flux boundary condition depends upon the properties (density, velocity) in the thin layer of gas between the propellant surface and the physical interface referred to as the combustion zone. The thickness of this combustion zone is assumed to be small, although much greater than the displacement of the physical interface. The velocity of the gas in the combustion zone is given by

$$U_2^* = \bar{c}\bar{M}_b n_0 + \epsilon u_2 e^{i\omega t} \tag{13}$$

The velocity is appropriate for end-burning T-burners, impedance tubes, and many theories.[12]

Further, it is assumed that all linear, unsteady (acoustic) responses of region 2 can be incorporated into the simple relationship

$$u_2 \cdot n = A_b p \qquad \text{on } S \tag{14}$$

These regions include acoustic propagation through the gas layer, fluctuations in the thermal layer of the solid, possible changes in the flame structure,[13] and other complex mechanisms.

The principal objection to the use of Eq. (14) is that such a "point/reacting" surface does not consider axial or parallel acoustic propagation in region 2. Therefore, this relationship is appropriate to acoustic duct liners, which physically restrict the acoustic propagation to a single, normal direction (an array of Helmholtz resonators), or to one-dimensional combustion experiments, such as an end-burning T-burner or impedance tube. It is noted that a complete removal of this

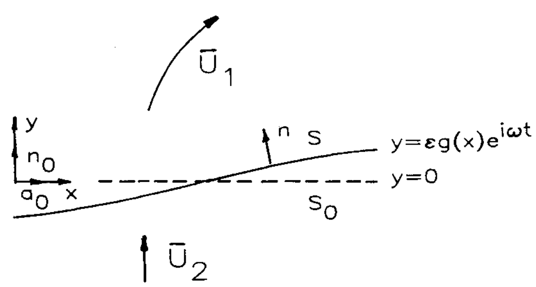

Fig. 3 Coordinate system for the acoustic boundary.

restriction would reduce the "stiffness" of region 2 and require a simultaneous solution of the acoustic equations in regions 1 and 2.[14] Since attempts to model the acoustic response of propellants to normal disturbances have not been very successful,[12] the development of a theoretical analysis of the response to parallel disturbances is not an immediate prospect. Thus, the development of an acoustical model involving a complete coupling of regions 1 and 2 for axial wave propagation remains a distant goal. However, Flandro[15] has developed an analysis to account for the damping of parallel oscillations by viscosity in the gaseous combustion zone. This analysis is useful in that it accounts for the effect of axial or parallel oscillations in the gaseous portion of region 2, while retaining the "stiff" form of Eq. (14). Thus, it is appropriate to interpret the admittance in Eq. (14) as the corrected admittance from Ref. 15.

It is also noted that Eq. (14) is written in the same form as the Flandro correction, i.e., it is an acoustic relation between the pressure and velocity of the gas in region 2 evaluated at the boundary. For an impenetrable surface, Myers defines Eq. (14) in terms of the velocity of the surface and the pressure in region 1, thus removing any dependence on the physical state of the material within region 2. Although the form of Eq. (14) is physically appropriate to the combustion admittance problem, it can be shown that both approaches yield identical results.

Motor Cavity Analysis

Within region 1 (the motor cavity), the velocity is

$$U_1^* = \bar{U}_1 + \epsilon u_1 e^{i\omega t} \tag{15}$$

For convenience, the steady velocity is written as the sum of a steady flux term and a steady nonflux term

$$\bar{U}_1 = \bar{c}\bar{M}_b n_0 + \bar{V}_1 \tag{16}$$

and it is assumed that the sound speed $\bar{c}$ is the same in both regions.

The boundary condition for the steady flux represents the requirement that mass must be conserved at the interface. Recalling that the undisturbed surface ($\epsilon = 0$) is equivalent to the mathematical boundary, conservation of mass requires that

$$\bar{\rho}_1 U_1 \cdot n_0 = \bar{\rho}_2 \bar{U}_2 \cdot n_0 \qquad \text{on } S_0 \tag{17}$$

Assuming the steady density to be continuous across the boundary ($\bar{\rho}_1 = \bar{\rho}_2$), Eqs. (13), (15), and (16) are substituted into Eq. (17) to show that

$$\bar{V}_1 \cdot n_0 = 0 \qquad \text{on } S_0 \tag{18}$$

Thus, the velocity $\bar{V}_1$ represents the nonflux steady flow in the motor cavity, which is composed of the steady cross flow parallel to the mathematical boundary and the effect of normal gradients that cause the normal flux to turn parallel to the axis.

The focus of this paper is the fluctuations in the mass flux at the stationary mathematical boundary that appear as an integrand in stability analyses.[1] The general form of this integrand is

$$m = \rho_1^* U_1^* \cdot n_0 - \bar{\rho}_1 \bar{c}_1 \bar{M}_b \qquad \text{on } S_0 \tag{19}$$

Substituting Eqs. (1), (15), and (16) into Eq. (19) and using Eq. (18) gives

$$m = [\bar{\rho}_1 u_1 \cdot n_0 + \bar{c}\bar{M}_b \rho_1] e^{i\omega t} \qquad \text{on } S_0 \tag{20}$$

Specifically, the term in brackets is the integrand appearing in the stability analysis of Ref. 1 and will be denoted by m_c. This expression must be related to the properties of region 2 (admittance, etc.) using the boundary condition at the physical interface S.

In the presence of disturbances, mass must be conserved at the physical interface S,

$$\rho_1^* U_1^* \cdot n = \rho_1^* U_2^* \cdot n \qquad \text{on } S \tag{21}$$

The normal to the physical interface can be obtained by writing Eq. (12) in the equivalent form

$$f(x,t) = y - \epsilon g(x) e^{i\omega t} = 0 \tag{22}$$

so that the unit normal vector is

$$n = \frac{\nabla f}{|\nabla f|} = n_0 - \epsilon \frac{\partial g}{\partial x} e^{i\omega t} a_0 \tag{23}$$

Note that in the rectangular coordinate system, the stationary unit normal vector n_0 does not change direction in moving from the surface S to S_0. The instantaneous unit normal to the surface S does, however, have an axial component.

Substituting Eqs. (1), (13), (15), (16), and (23) into Eq. (21) and neglecting terms of order ϵ^2 and higher gives

$$\bar{\rho}_1 \bar{V}_1 \cdot n_0 + \epsilon\left(\bar{\rho}_1 u_1 \cdot n_0 - \bar{\rho}_1 \bar{V}_1 \cdot a_0 \frac{\partial g}{\partial x} + \rho_1 \bar{c}\bar{M}_b + \bar{\rho}_1 \bar{V}_1 \cdot n_0\right) e^{i\omega t} = \epsilon(\bar{\rho} u_2 \cdot n_0 + \rho_2 \bar{c}\bar{M}_b) e^{i\omega t} \qquad \text{on } S \tag{24}$$

Each term in Eq. (24) must be evaluated at the mathematical boundary by expanding each term in a Taylor's series about S_0. For example,

$$\bar{\rho}_1 \bar{V}_1 \cdot n_0 \Big|_S = \bar{\rho} \bar{V}_1 \cdot n_0 \Big|_{S_0} + \epsilon \bar{\rho} \frac{\partial(\bar{V}_1 \cdot n_0)}{\partial y} \Big|_{S_0} g\, e^{i\omega t} \tag{25}$$

and the first term on the right-hand side of Eq. (25) is zero due to Eq. (18). Since the first term on the left-hand side of Eq. (24) is the only term not of order, ϵ, it is clear that only Eq. (25) provides a term of order ϵ, with all other expansion terms being of order ϵ^2 or higher. Thus, substitution of Eq. (25) into Eq. (24) and rearranging terms gives the integrand

$$m_c = \bar{\rho}\left(u_2 \cdot n_0 + \bar{V}_1 \cdot a_0 \frac{\partial g}{\partial x} - \frac{\partial \bar{V} \cdot n_0}{\partial y} g\right) + \rho_2 \bar{c}\bar{M}_b \quad \text{on } S_0 \tag{26}$$

where, as previously noted, the subscript c denotes removal of the exponential factor.

The physical interface is assumed to be defined by acoustic particles that are not disturbed by the steady mass flux, but are displaced by fluctuating forces. Using a theorem credited to Lagrange,[5] any acoustic particle on the boundary must remain on the boundary for all time. This requirement is expressed by

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}x_s}{\mathrm{d}t} \cdot \nabla f = 0 \tag{27}$$

From Eq. (23), it is clear that Eq. (27) requires the normal velocity component of the surface. The normal velocity of the surface may be obtained from the basic kinematic boundary condition that the normal velocity of the surface must be equal to the normal velocity in the gas adjacent to the surface.[5,11] Since the interface does not respond to the steady normal flux, the normal velocity of the surface must be equal to the sum of all normal velocity components of order ϵ. This kinematic boundary condition is

$$(\bar{V}_1 + \epsilon u_1 e^{i\omega t}) \cdot n = \epsilon \frac{\mathrm{d}x_s}{\mathrm{d}t} \cdot n = \epsilon u_2 e^{i\omega t} \cdot n \tag{28}$$

Substituting Eq. (28) into Eq. (27) and using Eq. (22) yields

$$g = iu_2 \cdot n/\omega \qquad \text{on } S \tag{29}$$

Substituting Eq. (14) into Eq. (29) and noting that no Taylor's series expansion is necessary produces the results

$$g = iA_b p_2/\omega \tag{30}$$

The steady flux across the physical boundary will carry acoustic momentum, so that it is not strictly correct to classify the physical boundary as a free surface. It can be argued that this flux is negligible for the low-speed steady flow considered here. (It has been assumed that the steady density and sound speed are constant.) In any event, it is consistent with the assumptions of Ref. 1 to employ the kinetic boundary condition[5]

$$\bar{p}_1 = \bar{p}_2$$

and

$$p_1 = p_2 \tag{31}$$

Acknowledging that the fluctuations at the flame are not isentropic, the approximation of Ref. 1 is retained

$$\bar{c}^2 \rho_2 = p_2 = p_1 \tag{32}$$

Substituting Eqs. (30) and (32) into Eq. (26) gives the desired integrand

$$m_c = \bar{\rho}_1 \left(A_b p_1 - \frac{iA_b \bar{U}_1 \cdot a_0}{\omega} \frac{\partial p_1}{\partial x} + \frac{ip_1 A_b}{\omega} \frac{\partial \bar{U}_1 \cdot n_0}{\partial y} \right) + \frac{\bar{M}_b}{\bar{c}} p_1 \qquad \text{on } S_0 \tag{33}$$

where it has been assumed that the admittance A_b is independent of axial location. The steady velocity $\bar{V}$ has been replaced by the more general $\bar{U}$ without any change in meaning.

The approach presented here is directed toward the combustion community. Only one boundary condition [Eqs. (21)] is specified, while Eqs. (28) and (31) represent further mathematical requirements that the physical interface is a free surface and that the two acoustic regions must not mix. From a rigorous mathematical approach, it would be proper to identify the basic kinematic boundary condition of fluid mechanics [Eq. (28)] and the kinetic boundary condition [Eq. (31)] as the boundary conditions and Eq. (21) as the specification of mass flux at the interface. Using either approach, the results would be identical. Reference 11 uses the latter approach for an impermeable surface and it is apparent that substitution of Eqs. (11) and (32) into Eq. (20) produces the integrand for a general curvilinear surface coordinate system

$$m_c = \bar{\rho} \left[A_b p_1 - \frac{iA_b \bar{U}_1}{\omega} \cdot \nabla p_1 + \frac{ip_1 A_b}{\omega} n_0 \cdot (n_0 \cdot \nabla \bar{U}_1) \right] + \frac{\bar{M}_b}{\bar{c}} p_1 \qquad \text{on } S_0 \tag{34}$$

where n_0 is the unit normal to S_0. For a rectangular coordinate system, Eq. (34) reduces to Eq. (33). Further, it can be shown that for a constant-area end-burning T-burner Eq. (34) reduces to the familiar expression

$$m_c = [\bar{\rho} A_b + (\bar{M}_b/\bar{c})] p_1 \qquad \text{on } S_0 \tag{35}$$

This expression is also used at the propellant surface in standard stability analyses[1] in place of Eq. (33).

Although there are two terms present in Eq. (33) that are not present in the integrand used for most stability analyses, the physical interpretation of their mechanism of energy transfer is similar. In this case, there is both a steady mass flux and an oscillating mass flux across the mathematical boundary, while there is only a steady mass flux across the physical interface. Thus, there is no oscillatory transfer of mass into region 1. The physical boundary of the motor cavity oscillates like a flexible piston and transmits acoustic energy by *p-v* work at the interface.

In stability analyses, it is common practice to separate the effects of mass fluctuations at the boundary representing the propellant surface into a coefficient of the pressure fluctuations (the pressure-coupled response function) and a coefficient of the axial velocity fluctuations (the velocity-coupled response function). For lack of a proper description, both response functions are taken to be properties of the propellant, although it has been pointed out that at least the velocity-coupled response function should depend on the velocity at the edge of the combustion zone.[16]

Within the combustion zone, the classification of response functions is based on the general form of the acoustic description for region 2. The limitations of a one-dimensional "point-reacting surface" described by Eq. (14) have been previously discussed. Such a response is familiar in combustion instability research and will be referred to as an omega response, since it can be characterized by the dimensionless variable Ω.[12] It is considered to be a property of the propellant and is commonly measured in T-burners and impedance tubes. The difficulties in formulating and applying an "extended reaction surface" analysis to region 2 have also been discussed. Such an analysis is required to describe properly the effect of a cross flow on the combustion zone. This effect will be referred to as erosive response,[16] since it is assumed to be an unsteady analog of erosive burning. It is not entirely a property of the propellant, but should depend on the cross flow at the edge of (and extending into) the combustion zone. It is generally assumed that the pressure coupling is a result of an omega response, while velocity coupling is due to an erosive response, although Price[16] has pointed out that this may not be absolute.

The first and last terms on the right-hand side of Eq. (33) represent the familiar omega pressure-coupled response terms.[1]

The second term in Eq. (33) occurs because the physical surface is not always parallel to the axis of the motor and there is a component of the steady axial velocity normal to the physical interface.

Using the acoustic axial momentum equation in region 1,

$$i\omega\bar{\rho}u_x = -\frac{\partial p}{\partial x}$$

the second term in Eq. (33) is written

$$-\frac{iA_b \bar{U}_x}{\omega} \frac{\partial p}{\partial x} = -\bar{\rho}\bar{U}_x A_b u_x$$

Thus, while only omega responses are considered, the boundary condition includes a velocity-coupling term that is dependent on the local steady flow in the motor and on the admittance characterized by omega. It is noted that this term does not address erosive velocity coupling, which is traditionally assumed to be the dominant velocity-coupling effect.

The third term on the right-hand side of Eq. (33) arises because the radial steady flow leaving the burning surface must either accelerate or decelerate (depending on the steady flow model) in order to turn toward the axial direction.

Writing the steady radial momentum equation at the boundary S_0 as

$$\frac{\partial \bar{U} \cdot n_0}{\partial y} = \frac{\partial \bar{U}_y}{\partial y} = -\frac{1}{\bar{c}\bar{M}_b} \frac{\partial p}{\partial y}$$

it can be seen that the steady pressure in the gas at the displaced boundary differs from the pressure at the undisturbed boundary, thus altering the p-v work done by the interface. This term is classified as an omega pressure-coupling term, but like the velocity-coupling term, it involves the imaginary part of the admittance and is dependent on the steady flow at the edge of the combustion zone.

Application to Rocket Motors

As a brief illustration of the effect of this formulation on the rocket stability problem, Eq. (33) is used to predict the growth coefficient of a simple rocket motor geometry. The configuration is the axisymmetric tubular grain illustrated in Fig. 4. For the motor-oriented coordinate system, the integrand can be written as

$$m_c = \bar{\rho}\left(A_b p - \frac{iA_b \bar{U}_z}{\omega}\frac{\partial p}{\partial z} + \frac{ipA_b}{\omega}\frac{\partial \bar{U}_r}{\partial r}\right) + \frac{\bar{M}_b}{\bar{c}}p \qquad (36)$$

on the surface $r = R_w$ and the subscript l can be omitted. The procedure for calculation of this decay coefficient is described in Refs. 1 and 17. The sign convention of Ref. 17 will be used so that, if Eq. (35) is used to represent the boundary condition at the burning propellant, the growth coefficient for the first axial mode is written as

$$\alpha = 4f(L/D)(A^r - \bar{M}_b) \qquad (37)$$

where $f = \bar{c}/2L$ and $\bar{c}$ is the sound speed.

Note that Eq. (35) is independent of the flow model used to describe the steady flow in the rocket motor. However, the expression for the decay coefficient obtained using the boundary condition given by Eq. (36) is strongly dependent on the steady flow description. Two steady flow descriptions will be considered. The first description is the "uniform" flow field given by

$$\bar{U}_z = (2\bar{M}_b z/R_w) \qquad \bar{U}_r = (-\bar{M}_b r/R_w) \qquad (38)$$

Substituting Eqs. (36) and (38) into the appropriate equations[1] and evaluating the integrals gives two additional growth terms

$$\alpha_{\mathrm{LVC}} = \frac{8fA_b^i\bar{M}_b}{\pi}\left(\frac{L}{D}\right)^2 \qquad (39)$$

from the linear omega velocity-coupling term and the same value

$$\alpha_{\mathrm{FA}} = \frac{8fA_b^i\bar{M}_b}{\pi}\left(\frac{L}{D}\right)^2 \qquad (40)$$

for the steady flow acceleration term. The total growth coefficient is

$$\alpha = 4f\left(\frac{L}{D}\right)\left[A_b^r + \frac{4A_b^i\bar{M}_b}{\pi}\left(\frac{L}{D}\right) - \bar{M}_b\right] \qquad (41)$$

The second flow description is presented in Ref. 18 as a "rotational" flowfield given by

$$\bar{U}_z = \frac{\pi\bar{M}_b z}{R_w}\cos\left(\frac{\pi r^2}{2R_w^2}\right)$$

$$\bar{U}_r = -\frac{\bar{M}_b R_w}{r}\sin\left(\frac{\pi r^2}{2R_w^2}\right) \qquad (42)$$

Substituting Eqs. (36) and (42) into the appropriate equations and evaluating the appropriate integrals gives

$$\alpha_{\mathrm{LVC}} = 0 \qquad (43)$$

and

$$\alpha_{\mathrm{FA}} = -\frac{8fA_b\bar{M}_b}{\pi}\left(\frac{L}{D}\right)^2 \qquad (44)$$

Note that Eq. (40) has the opposite sign to Eq. (44) since, for the uniform flow model, $\partial\bar{U}_r/\partial r = -\bar{M}_b/R_2$, indicating a deceleration of the steady flow normal to the burning surface, while for the rotational model $\partial\bar{U}_r/\partial r = \bar{M}_b/R_w$ at the surface, indicating an initial acceleration of the steady flow normal to the burning surface. Then total growth coefficient for the rotational steady flow model is

$$\alpha = 4f\left(\frac{L}{D}\right)\left[A_b^r - \frac{2A_b^i\bar{M}_b}{\pi}\left(\frac{L}{D}\right) - \bar{M}_b\right] \qquad (45)$$

Note that, if A_b^i is positive, Eq. (41) predicts an additional destabilizing influence, while Eq. (45) predicts an additional stabilizing term.

Equations (38) and (42) are useful in that they are closed-form descriptions of the steady flow in the motor cavity. Unfortunately, the two expressions predict opposite effects for the flow acceleration term. More definitive results could be obtained using the steady velocity gradient predicted by the NASTRAN potential flow codes in the AFRPL Standard Stability Prediction Codes.[19] It is also noted that primary emphasis has been placed on measuring values of the real part of the propellant admittance due to its importance in Eq. (35). Values of the imaginary part, which are important in Eq. (34), are seldom reported.[20]

Although the focus of this paper has been matching of the combustion zone to the motor cavity, this derivation is applicable to the exhaust nozzle. In practice, Eq. (35) is used to describe the influence of the nozzle on the motor cavity. Although the second term in Eq. (34) is unimportant, the third term is applicable for certain nozzle configurations. If the entrance to the nozzle is smooth (as illustrated in Fig. 4), Eq. (35) is appropriate. If the nozzle is submerged so that the steady flow is accelerating (or decelerating) entering the nozzle, the nozzle admittance used in Ref. 1 should be replaced by the "corrected nozzle" admittance

$$A_n + \frac{iA_n}{\omega}\frac{\partial\bar{U}_x}{\partial x}\bigg|_L$$

where A_n is the nozzle admittance for the equivalent nozzle with a smooth entrance that is often calculated from a one-dimensional theory. This correction may be important in understanding the failure of one-dimensional nozzle damping calculations to describe the damping of submerged nozzles.[21] It is noted that the axial steady velocity gradient can be computed from the NASTRAN potential flow code and the imaginary part of the nozzle admittance can be computed using one-dimensional[22] or two-dimensional[23] nozzle damping analyses.

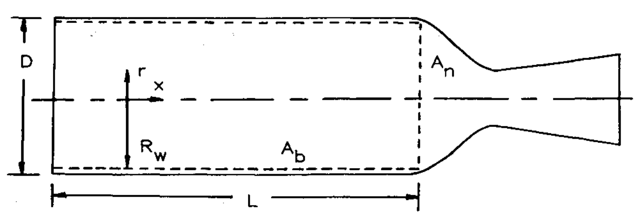

Fig. 4 Geometry of an axisymmetric rocket motor.

Discussion

The acoustic boundary condition at an admittance surface in the presence of a steady flow has been a subject of debate in the field of aeroacoustics for many years. A recent paper by Myers[11] presents a derivation of the boundary condition that avoids the debatable postulations of some earlier theories and arrives at an expression that is verified by limiting solutions and experiments. The derivation recalls the old adage that the time average of the products is not necessarily the product of the time averages. In this case, the time average of the boundary condition applied at the boundary is not in general equal to the time average of the boundary condition applied at the time-averaged boundary.

The boundary condition is written as the conservation of mass at the instantaneous interface (admittance surface) between the gas in the motor cavity and the gas in the thin combustion zone. The boundary condition is then transformed to the time-averaged interface (boundary) location for application to the linear acoustic problem. Although there appears to be both steady and unsteady mass flux across the time-averaged boundary, there is only a steady mass flux across the instantaneous boundary and acoustic energy is transmitted to the motor cavity by p-v work at the instantaneous boundary. This formulation introduces two additional terms not present in the simple admittance condition. These additional terms are due to mass flux created by discontinuities in the steady axial velocity and the steady radial velocity derivative at the admittance surface.

This formulation can be interpreted in two ways. If admittances obtained from one-dimensional theories or one-dimensional experiments (T-burner) are applied to rocket stability analyses, Eq. (34) should be employed as the boundary condition to account for the discontinuities in steady flow properties. Alternatively, if the simple admittance condition is preferred, two-dimensional theories and experiments should be developed that reproduce the tangential steady flow and steady flow gradient normal to the surface in the combustion zone.

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